Explaining the Solar Analema

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PHYS 3380: Astronomy

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HOW AND WHERE I OBSERVED THE SETTING SUN

From late-August to mid-November, I observed and noted the location of the setting sun with a roughly once-per-week frequency. I made these observations at UTD, near south side of Phase 3 (just north-northwest of the soccer fields). I selected this location for its relatively clear view of the western horizon and its saturation with landmarks (light poles, trees, a road, and a stop-sign). There is a shed that Facilities uses to store lawn equipment and a gravel ramp of ~ five foot elevation that leads to the shed. I stood on the highest point of this ramp when I made all my observations.

While the view of sky from this location extended quite close to the true horizon, a distant line of apartment buildings covered the last ~half-degree (30') of the sky. Since $\frac{1}{2}°$ is the angular size of the sun, and since the sun travels at $\frac{15°}{1 \text{ hour}} = \frac{.25°}{1 \text{ minute}}$, my "sunset" was offset by ~two minutes before true sunset. However, since this line of buildings affected all of my observations uniformly, this premature measure of the sunset is not a source of error (as I will discuss in the final section of this report).

Aside from this minor complication, my observations were straightforward: I simply drew where I saw the sun with respect to the landmarks near the horizon. The apparent angular diameter that I drew for the sun reflects my uncertainty of its position due to the phenomenon of *irradiation*, the "washing out" of the retina due to the brightness of the sun (A similar effect occurs when taking a picture of the sun or when taking a long-exposure picture of a bright star or planet).

For the sake of rigor, I wanted to also know how large the sun would appear if the human eye could successfully make out its disk without being oversaturated. So on September 12th, I used shade 14 welder's glass (a highly opaque piece of glass that permits the transmission of only the brightest light—magnitudes $m < -26$—and removes the phenomenon of *irradiation*) and drew what I saw through the glass (you can see the faintly drawn smaller circle on my entry "9/12; 7:35 p.m.). The sun's disk ended up being 1 cm in diameter, revealing the significant error in my 2 cm-in-diameter drawings of the Sun including irradiation due to naked-eye observation. Please refer to the final section of this report for a discussion on whether the drawings including the irradiation serve as a significant source of error.

HOW AND WHY THE SUNSET POSITION CHANGES WITH RESPECT TO TIME

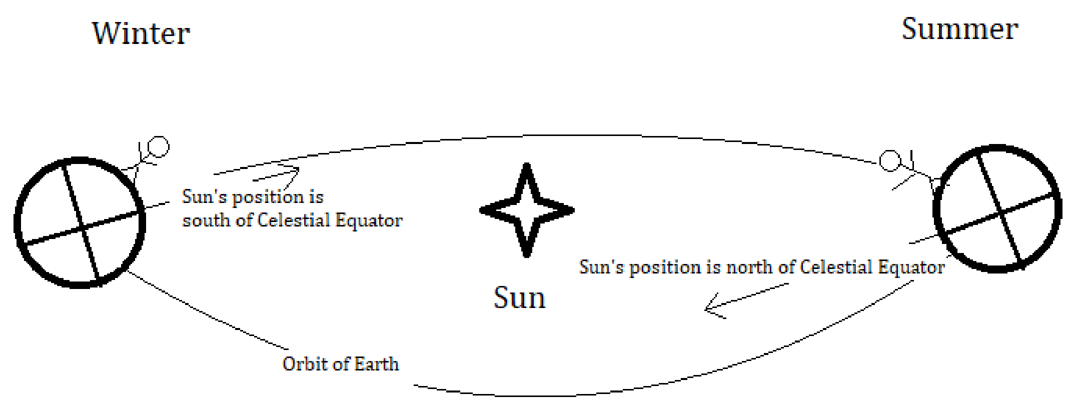
My first observations were made as summer was beginning to wane. These late-August to early-September observations showed the Sun's position to be north of due west by about 12° on

August 27th and about 11° on September 2nd (see observations attached, noting again that my apparent angular diameter of the sun, which includes *irradiation*, does not equal the Sun's actual angular diameter of $\frac{1}{2}$°).

As the weeks went on, the sunset position continued to approach due west, finally crossing that point precisely when we would expect it: on some date between September 15th and September 25th (Note that these dates average nicely to September 20th, which is very close to the autumnal equinox of September 21st).

After this date, the setting Sun was found only south of due west. The Sun's position became more southerly we approached winter, most recently reaching 19° on November 14th and 22° on November 20th.

We can use a simple Earth-Sun diagram to rationalize why the sunset position changes with respect to time:

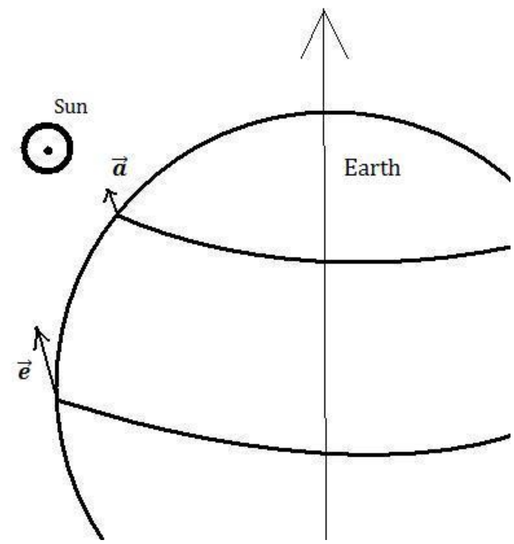


The geometry shows us that in the summer, the Sun is observed to be north of the Celestial Equator (and hence in the northern sky), while in the winter, the Sun is observed to be south of the Celestial Equator (and hence in the southern sky).

The above explanation is very intuitive. However, I would like to be more analytical and find an expression for the exact angle by which the sunset position deviates from due west as a function of time.

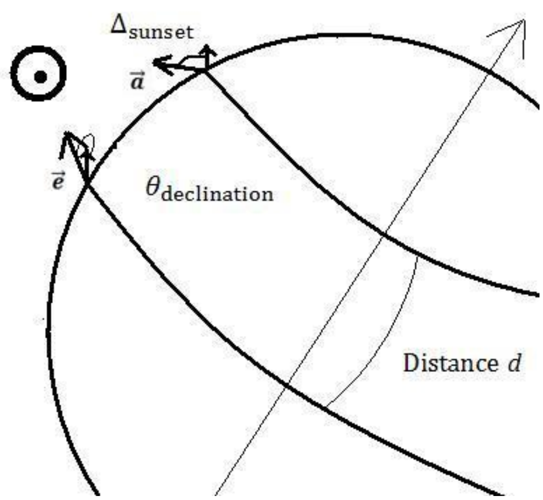
We can approach this task by drawing a simple diagram that reveals the crucial fact: the declination $\theta_{\text{declination}}$ of the Sun *equals* the angle by which the sunset position deviates from due west Δ_{sunset}.

On the equinoxes, the spin axis of the earth (marked below by the long, thin arrow) is tilted neither toward nor away from the sun. So at sunset on the equinoxes, the arrows $\vec{a}$ and $\vec{e}$ point due east towards the distant sun:



The angle subtended by arrow $\vec{\boldsymbol{e}}$ and the Celestial Equator is defined as the declination of the sun $\theta_{\text{declination}}$. On the equinoxes, the declination of the sun is 0°. The angle subtended by arrow $\vec{\boldsymbol{a}}$ and the Celestial Equator is the north-south angle from due west to the Sun at sunset Δ_{sunset} . This diagram shows us that $\vec{\boldsymbol{a}} = \vec{\boldsymbol{e}}$ on the equinoxes, which implies that $\theta_{\text{declination}} = \Delta_{\text{sunset}}$.

The question as to *what happens on non-equinox days* arises. Is $\theta_{\text{declination}} = \Delta_{\text{sunset}}$ on non-equinox days, too? Let's draw a similar diagram with the Earth oriented as it is in winter, with the spin axis tilted away from the Sun:



In this diagram, $\theta_{\text{declination}}$ does not appear to be equal to Δ_{sunset}. But this is just the consequence of our perspective, with earth drawn close-up and the sun drawn off in the distance. If you consider that the distance d between the two latitudes is much, much less than the distance from the Earth to the Sun (that is, consider this diagram in the limit that $d \ll 1$ AU), then the angles would indeed appear be equal.

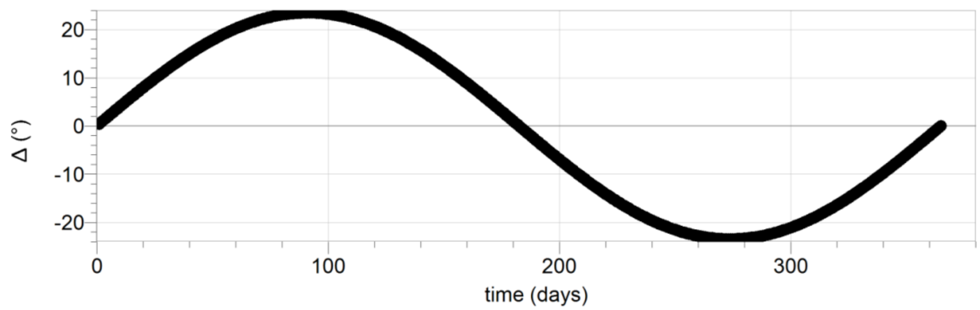
So in general, $\theta_{\text{declination}} = \Delta_{\text{sunset}}$!

$\theta_{\text{declination}}$ of the sun is an easily derivable function of time. Considering that it's a sinusoid whose amplitude is the obliquity of the earth and whose period is 365 days, we get

$$\theta_{\text{declination}} = \omega \sin\left(\frac{2\pi t}{365}\right) = \Delta_{\text{sunset}}$$

Note that ω is the obliquity of Earth $\approx 23.5°$ and that t is time in days where $t = 0$ is set as March 21st.

$$\Delta_{\text{sunset}}(t) = 23.5 * \sin\left(\frac{2\pi t}{365}\right)$$

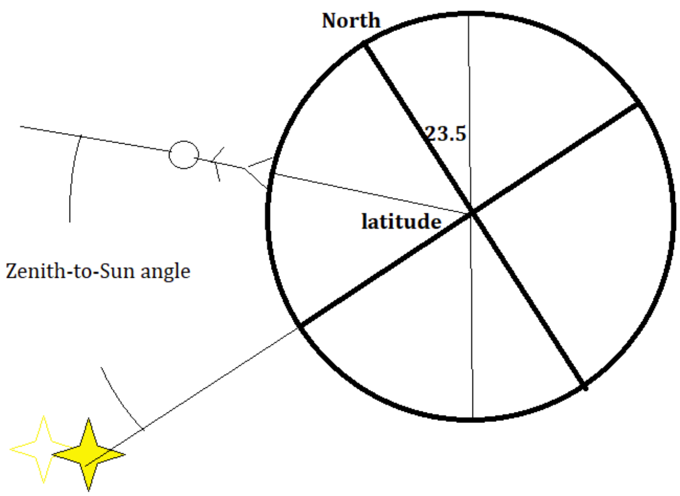


We can use this to predict the angle by which the sunset position deviates from due west.

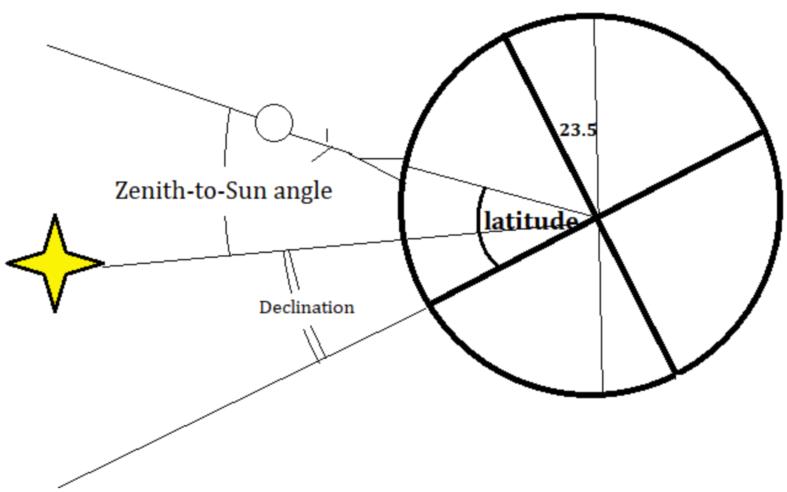
~Aside~

In the process of deriving this straightforward equation, I was also able to find the angle from zenith to the Sun's position at high noon as a function of time in days. This is a slightly more complicated scenario because the latitude of the observer becomes relevant. Let's again use a few simple diagrams to find the zenith-to-Sun angle on any given day of the year at any latitude:

On the equinoxes, the zenith-to-sun angle Δ_{zenith} equals the latitude of the observer θ_{latitude}:



A non-equinox day (the Earth-Sun geometry of summer is drawn below) requires a more thought:



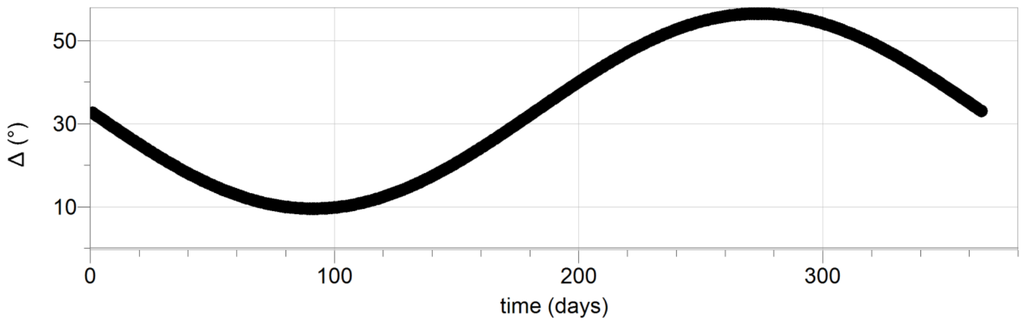
One can see the simple relationship that the sum of the zenith-to-sun and declination angles equals the latitude of the observer. Rearranging for Δ_{zenith}, we get $\Delta_{\text{zenith}} = \theta_{\text{latitude}} - \theta_{\text{declination}}$.

We already found that the declination of the sun is $\theta_{\text{declination}} = \omega \sin\left(\frac{2\pi t}{365}\right)$. So we get

$$\Delta_{\text{zenith}} = \theta_{\text{latitude}} - \omega \sin\left(\frac{2\pi t}{365}\right)$$

where Δ_{zenith} is the angle from zenith to the Sun's position at high noon, ω is the obliquity of Earth $\approx 23.5°$, t is time in days where $t = 0$ is set as March 21st, and θ_{latitude} is the observer's latitude on Earth (~33° for Richardson). Substituting in the known values, we get

$$\Delta_{\text{zenith}}(t) = 23.5 * \sin\left(\frac{2\pi t}{365}\right) - 33°$$



Many cultures have long been aware of this relationship, erecting monuments to celebrate $\Delta_{\text{zenith}}(t)$'s maxima, minima, and inflection points.

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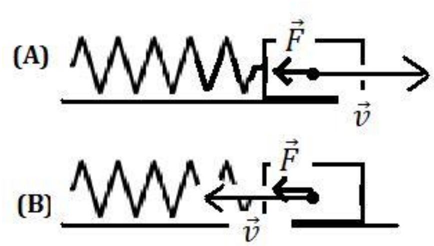
HOW AND WHY THE RATE AT WHICH THE SUNSET POSITION CHANGES OVER TIME

My observations show that the sunset position over the course of days changes at a non-uniform rate. Specifically, the sunset position changed most rapidly around the autumnal equinox and changed most slowly in late-August and mid-November (close to the solstices).

I must note that due to weather conditions, the frequency with which I observed the setting sun was not exactly once per week. Some consecutive observations are only a two to three days apart, while others are more than a week apart. This irregular method of data collection distorts the visual trend of the non-uniformity of the sunset position that would otherwise be clearly observed.

The non-uniformity of the sunset position can be rationalized by considering that the sunset position attains a maximum on minimum on the summer and winter solstices respectively. That is, on the solstices, the direction the sunset position moves *changes*. This change is continuous, analogous to the change in direction a spring undergoes between positions **(A)** and **(B)**:

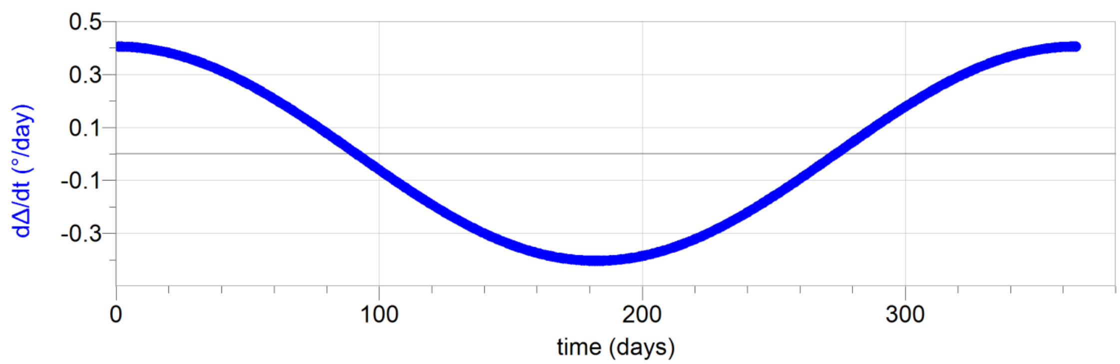


This analog that compares the block to the sun reveals a profound principle: For a differentiable (and hence continuous) path to undergo a change in direction, some continuous acceleration *must* be involved.

Obviously, the non-constant rate at which the sunset position changes over time is not actually due to the Sun being accelerated in space; rather, our perspective makes it look like the Sun is oscillating back and forth about some equilibrium point (the equinox). The apparent acceleration acting on the Sun is a consequence of the interaction of two misaligned coordinate systems (the plane of the solar system and the plane defined by Earth's equator).

We can very easily find an expression for the rate at which the sunset position changes by simply taking the derivative of $\Delta_{\text{sunset}}(t)$ with respect to time. The fact that $\frac{d}{dt}\Delta_{\text{sunset}}(\text{t}) \neq 0$ also verifies that the change in the sunset position is non-uniform.

$$\frac{d}{dt}\Delta_{\text{sunset}}(\text{t}) = \frac{d}{dt}\left[23.5 * \sin\left(\frac{2\pi t}{365}\right)\right]$$
$$= \frac{2\pi}{365} * 23.5 * \cos\left(\frac{2\pi t}{365}\right)$$



The plot above matches the fact that the rate at which the sunset position changes is maximum in magnitude on the equinoxes and is minimum on the solstices.

PROBLEMS & SOLUTIONS

As mentioned in the first section of this report, a distant line of apartment buildings covered the last ~half-degree (30') of the sky, offsetting my observations by ~two minutes before true sunset. I already discussed that the buildings were not a source of error because they affected all of my observations uniformly. In other words, my observations would match the observations of the true sunset made by someone who was $\frac{1}{2}°$ east of where I observed (equivalent to 34.5 miles east of UTD). So the main issue this discrepancy presents is not in the data but in the act of reporting my observations: it is misleading if I claim that my observations are of the "true sunset" when they are displaced by two minutes. I tried to solve this problem by adding a dashed line to my observation sheet labeled "Buildings."

One potential source of error is *irradiation*. Our naked eyes are too saturated with sunlight to accurately log the Sun's position in relation to landmark objects on the horizon. I tried to solve this problem by drawing circles that envelop the Sun's glare as well as its photosphere. But since our eyes perceive glare in a circular distribution around the light source, it is safe to pinpoint the location of the Sun itself by picking the center of the circles.

One final (and perhaps most valid) source of error was that I observed the Sun perhaps an additional two to three minutes before sunset on Wednesday, October 10th at 6:55 p.m., Friday, November 2nd at 6:32 p.m., and Tuesday, November 20th at 5:20 p.m. (see the observations highlighted in pink). When the Sun sets (especially around a solstice), its motion has a significant horizontal component. By making these few observations just a few minutes prior to when I usually recorded the sunset, I may have blurred a trend (namely, the non-uniformity with which the sunset position changes). Since it is hard to make out this non-uniformity from my data, I chose to supplement my report with the finding that $\frac{d}{dt}\Delta_{\text{sunset}}(\text{t}) \neq 0$.