Fourier acoustics[*][†]

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Mathematical preliminary

Recall the 2D Fourier transform inverse pair:

$$\hat{f}(k_x, k_y) = \mathcal{F}[f(x,y)] = \iint\limits_{-\infty}^{\infty} f(x,y) e^{-i(k_x x + k_y y)} dx dy \tag{1}$$

$$f(x,y) = \mathcal{F}^{-1}[f(k_x,k_y)] = \frac{1}{(2\pi)^2} \iint\limits_{-\infty}^{\infty} f(k_x,k_y) e^{i(k_x x + k_y y)} dk_x dk_y \tag{2}$$

Applying the inverse 2D Fourier transform to the n^{th} derivative of $f(x,y)$ with respect to x gives

$$\frac{\partial^n}{\partial x^n} f(x,y) = \frac{\partial^n}{\partial x^n} \mathcal{F}^{-1}[f(k_x,k_y)] = \mathcal{F}^{-1}[(ik_x)^n \hat{f}(k_x,k_y)]. \tag{3}$$

[*]based on Dr. Mark F. Hamilton's Acoustics II lecture on the topic, but in the $e^{i(kx-\omega t)}$ time convention

[†]to sort out my own understanding of the topic

Taking the Fourier transform of both left- and right-hand-sides of equation (3) results in

$$\mathcal{F}\left[\frac{\partial^n f(x,y)}{\partial x^n}\right] = \mathcal{F}\mathcal{F}^{-1}\left[(ik_x)^n \hat{f}(k_x,k_y)\right].$$

Since $\mathcal{F}$ and $\mathcal{F}^{-1}$ are inverses, the above equation results two identities (the second identity for derivatives with respect to y follows similarly):

$$\mathcal{F}\left[\frac{\partial^n f(x,y)}{\partial x^n}\right] = (ik_x)^n \hat{f}(k_x,k_y) \qquad \text{(ID 1)}$$

$$\mathcal{F}\left[\frac{\partial^n f(x,y)}{\partial y^n}\right] = (ik_y)^n \hat{f}(k_x,k_y) \qquad \text{(ID 2)}$$

Pressure source

The Helmholtz equation is the wave equation for time-harmonic solutions. In Cartesian coordinates,

$$0 = \nabla^2 p + k^2 p = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)$$

Taking the 2D spatial Fourier transform of the Helmholtz equation and applying (ID 1) and (ID 2) gives

$$\begin{aligned} 0 &= \mathcal{F}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right] \\ &= \left(-k_x^2 - k_y^2 - \frac{\partial^2}{\partial z^2} + k^2\right)\hat{p}(k_x,k_y,z) \end{aligned} \qquad (4)$$

But the orthogonality of the coordinate system relates the wavenumbers by $k^2 = k_x^2 + k_y^2 + k_z^2$. Thus equation (4) becomes an ordinary differential equation,

$$\left(\frac{d^2}{dz^2} + k_z^2\right)\hat{p}(k_x, k_y, z) = 0, \tag{5}$$

whose solution for $+z$-propagation is

$$\hat{p}(k_x, k_y, z) = \hat{p}_0(k_x, k_y)e^{ik_z z} \tag{6}$$

where

$$\begin{aligned}\hat{p}_0(k_x, k_y) &= \hat{p}_0(k_x, k_y, z = 0) \\ &= \mathcal{F}[p(x, y, z = 0)]\end{aligned}$$

is the source condition. The solution to the Helmholtz equation is found by applying equation (2) to equation (6):

$$p(x, y, z) = \mathcal{F}^{-1}\{\hat{p}_0(k_x, k_y)e^{ik_z z}\} \tag{7}$$

Equation (7) is equivalent to the second Rayleigh integral.

Velocity source

For velocity sources, start by recalling the linear momentum equation for time-harmonic solutions,

$$\boldsymbol{u} = \frac{1}{ik\rho_0 c_0}\boldsymbol{\nabla} p,$$

and apply the Fourier transform pair, equations (1) and (2),

$$\begin{aligned}\boldsymbol{u} &= \frac{1}{ik\rho_0 c_0}\boldsymbol{\nabla}\{\mathcal{F}^{-1}[\mathcal{F}p(x, y, z)]\} \\ &= \frac{1}{ik\rho_0 c_0}\mathcal{F}^{-1}\{\mathcal{F}[\boldsymbol{\nabla} p(x, y, z)]\} \qquad (8)\end{aligned}$$

Note from (ID 1) and (ID 2) for $n = 1$ that

$$\mathcal{F}\left[\frac{\partial p}{\partial x}\right] = ik_x\hat{p}$$
$$\mathcal{F}\left[\frac{\partial p}{\partial y}\right] = ik_y\hat{p}$$

Also note that $\frac{\partial}{\partial z}\mathcal{F}(p) = ik_z\hat{p}$ from equation (5). Then equation (8) becomes

$$\boldsymbol{u} = \frac{1}{\rho_0 c_0}\mathcal{F}^{-1}\left\{\frac{\boldsymbol{k}}{k}\hat{p}\right\} \tag{9}$$

The z-component of equation (9) becomes

$$u_z(x, y, z = 0) = u_0(x, y) = \frac{1}{\rho_0 c_0}\mathcal{F}^{-1}\left\{\frac{k_z}{k}\hat{p}(k_x, k_y)\right\} \tag{10}$$

Equation (10) for p_0 reads

$$\hat{p}_0(k_x, k_y) = \rho_0 c_0 \frac{k}{k_z}\hat{u}_0(k_x, k_y). \tag{11}$$

Substituting equation (11) into equation (7) gives

$$p(x, y, z) = \rho_0 c_0 \mathcal{F}^{-1}\left[\frac{k}{k_z}\hat{u}_0(k_x, k_y)e^{ik_z z}\right] \tag{12}$$
$$= \rho_0 c_0 \mathcal{F}^{-1}\left\{\mathcal{F}[u_0(x, y)]\frac{k}{k_z}e^{ik_z z}\right\} \tag{13}$$

Equation (13) is equivalent to first Rayleigh integral.