Guided electromagnetic waves [1]

Arbitrary waveguide. The governing equations are Maxwell's homogeneous equations, where $\boldsymbol{B} = \mu \boldsymbol{H}$, to preserve the form in [2] and [3].

$$\operatorname{div} \boldsymbol{E} = 0 \tag{1}$$

$$\operatorname{div} \boldsymbol{H} = 0 \tag{2}$$

$$\operatorname{curl} \boldsymbol{E} = -\mu \frac{\partial \boldsymbol{H}}{\partial t} \tag{3}$$

$$\operatorname{curl} \boldsymbol{H} = \epsilon \frac{\partial \boldsymbol{E}}{\partial t} \tag{4}$$

In a waveguide that is a perfect conductor, $\boldsymbol{E} = 0$ and $\boldsymbol{H} = 0$ inside the material, so the boundary conditions at the inner wall are

$$\boldsymbol{E}^{\parallel} = 0 \tag{5}$$

$$\boldsymbol{H}^{\perp} = 0 \tag{6}$$

The goal is to find a solution of the form

$$\boldsymbol{E} = [E_x(x,y)\hat{\boldsymbol{x}} + E_y(x,y)\hat{\boldsymbol{y}} + E_z(x,y)\hat{\boldsymbol{z}}]e^{i(kz-\omega t)} \tag{7}$$

$$\boldsymbol{H} = [H_x(x,y)\hat{\boldsymbol{x}} + H_y(x,y)\hat{\boldsymbol{y}} + H_z(x,y)\hat{\boldsymbol{z}}]e^{i(kz-\omega t)} \tag{8}$$

that satisfy equations (1), (2), (7), and (8). Putting equation (7) into equation (3) results in

$$\frac{\partial E_z}{\partial y} - ikE_y = i\mu\omega H_x \tag{9}$$

$$ikE_x - \frac{\partial E_z}{\partial x} = i\mu\omega H_y \tag{10}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\mu\omega H_z, \tag{11}$$

and putting equation (8) into equation (4) results in

$$\frac{\partial H_z}{\partial y} - ikH_y = -i\epsilon\omega E_x \tag{12}$$

$$ikH_x - \frac{\partial H_z}{\partial x} = -i\epsilon\omega E_y \tag{13}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = -i\epsilon\omega E_z, \tag{14}$$

Combining equations (10) and (12) gives E_x and H_y; combining equations (9) and (13) gives E_y and H_x.

$$E_x = \frac{i}{\epsilon\mu\omega^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \mu\omega\frac{\partial H_z}{\partial y}\right) \tag{15}$$

$$E_y = \frac{i}{\epsilon\mu\omega^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \mu\omega\frac{\partial H_z}{\partial x}\right) \tag{16}$$

$$H_x = \frac{i}{\epsilon\mu\omega^2 - k^2}\left(k\frac{\partial H_z}{\partial x} - \epsilon\omega\frac{\partial E_z}{\partial y}\right) \tag{17}$$

$$H_y = \frac{i}{\epsilon\mu\omega^2 - k^2}\left(k\frac{\partial H_z}{\partial y} + \epsilon\omega\frac{\partial E_z}{\partial x}\right) \tag{18}$$

Inserting equations (15) and (16) into equation (11) leads to equation (19), and inserting equations (17) and (18) into equation (14) leads to equation (20):

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \epsilon\mu\omega^2 - k^2\right)E_z = 0 \tag{19}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \epsilon\mu\omega^2 - k^2\right)H_z = 0 \tag{20}$$

- Transverse electric (TE) waves arise when $E_z = 0$.
- Transverse magnetic (TM) waves arise when $H_z = 0$.

- $E_z = 0$ and $H_z = 0$ (TEM) cannot simultaneously be 0 in a hollow waveguide. See Griffiths for the proof.

TE waves in a rectangular waveguide. For TE waves, equation (19) is identically 0, and equation (20) is solved by separation of variables, choosing $H_z = X(x)Y(y)$ as the basis function:

$$\frac{X''}{X} + \frac{Y''}{Y} + \epsilon\mu\omega^2 - k^2 = 0$$

Introducing the separation constants k_x^2 and k_y^2 such that $k^2 = \epsilon\mu\omega^2 - k_x^2 - k_y^2$, harmonic solutions are found:

$$\begin{aligned} X(x) &= A\sin(k_x x) + B\cos(k_x x) \\ Y(y) &= C\sin(k_y y) + D\cos(k_y y) \end{aligned}$$

The waveguide has transverse dimensions a and b, so by equation (6), the boundary conditions are $H_x(x = 0) = H_x(x = a) = H_y(y = 0) = H_y(y = b) = 0$. By equation (17),

$$\left.\frac{\partial H_z}{\partial x}\right|_{x=0} = 0 = \left.\frac{\partial H_z}{\partial x}\right|_{x=a}.$$

The first equality sets $A = 0$, and the second equality sets $k_x = m\pi/a$. Similarly, by equation (18),

$$\left.\frac{\partial H_z}{\partial y}\right|_{y=0} = 0 = \left.\frac{\partial H_z}{\partial y}\right|_{y=b}.$$

The first equality sets $C = 0$, and the second equality sets $k_y = n\pi/b$. Therefore, the solution $H_z = X(x)Y(y)$ becomes[1]

$$H_z(x, y) = H_0 \cos(m\pi x/a)\cos(n\pi y/b) \tag{21}$$

[1] $m, n \in 0, 1, 2, \ldots$ but m and n cannot be 0 simultaneously. If they were, $\oint E \cdot dl = -\mu ab\frac{\partial H_z}{\partial t} = 0$ by equation (5), and thus $H_z = 0$. Recall that $E_z = 0$ too. But TEM modes cannot occur in a hollow waveguide.

and the wavenumber is

$$k = \sqrt{(\epsilon\mu\omega^2)^2 - (m\pi/a)^2 - (n\pi/a)^2} \tag{22}$$

By equations (15), (16), (17), (18), the full electromagnetic field in the waveguide is

$$\boldsymbol{E} = [E_x(x,y)\hat{\boldsymbol{x}} + E_y(x,y)\hat{\boldsymbol{y}} + E_z(x,y)\hat{\boldsymbol{z}}]e^{i(kz-\omega t)} \tag{23}$$
$$\boldsymbol{H} = [H_x(x,y)\hat{\boldsymbol{x}} + H_y(x,y)\hat{\boldsymbol{y}}]e^{i(kz-\omega t)}, \tag{24}$$

where

$$E_x(x,y) = \frac{i\mu\omega}{\epsilon\mu\omega^2 - k^2}\left(\frac{n\pi}{b}\right) H_0 \cos(m\pi x/a)\cos(n\pi y/b) \tag{25}$$
$$E_y(x,y) = -\frac{i\mu\omega}{\epsilon\mu\omega^2 - k^2}\left(\frac{m\pi}{a}\right) H_0 \cos(m\pi x/a)\cos(n\pi y/b) \tag{26}$$
$$H_x(x,y) = \frac{ik}{\epsilon\mu\omega^2 - k^2}\left(\frac{m\pi}{a}\right) H_0 \cos(m\pi x/a)\cos(n\pi y/b) \tag{27}$$
$$H_y(x,y) = \frac{ik}{\epsilon\mu\omega^2 - k^2}\left(\frac{n\pi}{b}\right) H_0 \cos(m\pi x/a)\cos(n\pi y/b) \tag{28}$$
$$H_z(x,y) = H_0 \cos(m\pi x/a)\cos(n\pi y/b)\,, \tag{29}$$

and where the wavenumber k is given by equation (31).

TM waves in a rectangular waveguide. For TM waves, equation (20) is identically 0, and equation (19) is solved by separation of variables, choosing $E_z = X(x)Y(y)$ as the basis function. The general solution is of the same form as for the TE case above,

$$X(x) = A\sin(k_x x) + B\cos(k_x x)$$
$$Y(y) = C\sin(k_y y) + D\cos(k_y y)\,,$$

but they are particularized to the boundary condition given by equation (5):

$$E_z(x=0) = E_z(x=a) = E_z(y=0) = E_z(y=b) = 0$$

. The first and second equalities respectively set $B = 0$, $D = 0$, and the third and fourth equalities respectively set $k_x = m\pi/a$ and $k_y = n\pi/b$, where $m, n \in \{1, 2, \dots\}$.[2] Therefore, the solution $E_z = X(x)Y(y)$ becomes

$$E_z(x, y) = E_0 \sin(m\pi x/a) \sin(n\pi y/b) \tag{30}$$

and the wavenumber is

$$k = \sqrt{(\epsilon\mu\omega^2)^2 - (m\pi/a)^2 - (n\pi/b)^2} \tag{31}$$

By equations (15), (16), (17), (18), the full electromagnetic field in the waveguide is

$$\boldsymbol{E} = [E_x(x, y)\hat{\boldsymbol{x}} + E_y(x, y)\hat{\boldsymbol{y}} + E_z(x, y)\hat{\boldsymbol{z}}]e^{i(kz-\omega t)} \tag{32}$$

$$\boldsymbol{H} = [H_x(x, y)\hat{\boldsymbol{x}} + H_y(x, y)\hat{\boldsymbol{y}}]e^{i(kz-\omega t)} \tag{33}$$

where

$$E_x(x, y) = \frac{ik}{\epsilon\mu\omega^2 - k^2}\left(\frac{m\pi}{a}\right) E_0 \sin(m\pi x/a) \sin(n\pi y/b) \tag{34}$$

$$E_y(x, y) = \frac{ik}{\epsilon\mu\omega^2 - k^2}\left(\frac{n\pi}{b}\right) E_0 \sin(m\pi x/a) \sin(n\pi y/b) \tag{35}$$

$$E_z(x, y) = E_0 \sin(m\pi x/a) \sin(n\pi y/b) \tag{36}$$

$$H_x(x, y) = -\frac{ik\epsilon\omega}{\epsilon\mu\omega^2 - k^2}\left(\frac{n\pi}{b}\right) E_0 \sin(m\pi x/a) \sin(n\pi y/b) \tag{37}$$

$$H_y(x, y) = \frac{ik\epsilon\omega}{\epsilon\mu\omega^2 - k^2}\left(\frac{m\pi}{a}\right) E_0 \sin(m\pi x/a) \sin(n\pi y/b) \tag{38}$$

[2] $m \neq 0$ and $n \neq 0$ because these will make E_z vanish. Recall that $H_z = 0$ too. But TEM modes cannot simultaneously be 0 in a hollow waveguide.

References

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