Practice problem

In protest of the Acoustics I final exam historically being held on the last day of finals week, Dr. Hamilton's students stage a demonstration in front of the UT Administration Building. Equipped with megaphones, the students angrily voice their discontent with the final exam schedule.

As the demonstration wears on, the students yell louder and louder, to the point that Dr. Hamilton can hear them jeering from his office in ETC. Dr. Hamilton knows he must pacify his students, so as not to be held responsible for the demonstration and complicit in the students' angry messages. He bolts down the ETC stairs and runs toward the sound of the yelling.

A few minutes later, Dr. Hamilton finds his students in front of the Administration Building. He frantically takes off his shoes and socks and stuffs one of his socks down the mouth of a student's megaphones.

1. The megaphones have an exponential cross-sectional area, and the socks act as a rigid end to the mouth of the horn. The throat of the horn is a pressure-release boundary. Find the normal modes and eigenfrequencies for plane-wave motion in the sock-stuffed horn by separation of variables.

   Note that the boundary conditions are

   $$p(0) = 0 \ldots \text{pressure release}$$
   $$\left.\frac{\partial p}{\partial x}\right|_{x=L} = 0 \ldots \text{rigid end}$$

   The wave equation for the exponential horn reads

   $$p_{xx} + m p_x - \frac{1}{c_0^2} p_{tt} = 0 \tag{1}$$

   Assuming the basis for the solution is $p(x,t) = X(x)T(t)$, equation (1) becomes (after dividing by the basis)

   $$\frac{X''}{X} + m\frac{X'}{X} - \frac{1}{c_0^2}\frac{T''}{T} = 0 \tag{2}$$

   Solving equation (2) for $\frac{T''}{T}$, we introduce a separation constant $-k^2$ in anticipation of time-harmonic solutions:

   $$\frac{T''}{T} = -c_0^2 k^2 \tag{3}$$

   Identifying $\omega = c_0 k$, the solution for $T(t)$ is

$$T(t) = Be^{\pm j\omega t} \tag{4}$$

Now solving equation (2) for the functions of x,

$$\frac{X''}{X} + m\frac{X'}{X} = -k^2$$

Rearranging,

$$X'' + mX' + k^2 X = 0$$

Assuming the solution has the form $X = e^{rx}$, substituting into the above, and canceling the common exponential factors, gives the following characteristic quadratic equation:

$$\begin{aligned} r^2 + mr + k^2 = 0 \\ \implies r &= -\frac{m}{2} \pm \frac{1}{2}\sqrt{m^2 - 4k^2} \\ &= -\frac{m}{2} \pm j\sqrt{k^2 - \left(\frac{m}{2}\right)^2} \end{aligned}$$

$X(x)$ is therefore

$$\begin{aligned} X &= \exp\left(-\frac{mx}{2} \pm j\sqrt{k^2 - \left(\frac{m}{2}\right)^2}\right) \\ &= \exp\left(-\frac{mx}{2}\right)\left(A_1 \cos\left(\sqrt{k^2 - \left(\frac{m}{2}\right)^2} x\right) + A_2 \sin\left(\sqrt{k^2 - \left(\frac{m}{2}\right)^2} x\right)\right) \end{aligned}$$

The second line is due to the fact that linear combinations of complex exponentials give cosine and sine. Applying the boundary condition at $x = 0$ shows that $A_1 = 0$, and applying the boundary condition at $x = L$ gives

$$\left.\frac{\partial p}{\partial x}\right|_{x=L} = 0 = B\sqrt{k^2 - \left(\frac{m}{2}\right)^2}\cos\sqrt{k^2 - \left(\frac{m}{2}\right)^2}L = 0$$

$$\implies \sqrt{k_n^2 - \left(\frac{m}{2}\right)^2} = \frac{\pi}{2}(2n+1),\ n = \{0, 1, 2\ldots\} \quad \text{(Equation for } k_n\text{)}$$

The eigenfunctions are therefore

$$p_n(x,t) = \sum_{n=0}^{\infty} B_n \exp\left(-\frac{mx}{2}\right)\sin\left(\frac{\pi}{2L}(2n+1)x\right)e^{\pm j\omega_n t}$$

Solving the (Equation for k_n) for the eigenfrequencies f_n,

$$f_n = \frac{c_0}{2\pi}\sqrt{\left(\frac{\pi}{L}(n+1)\right) + \left(\frac{m}{2}\right)^2},\ n = \{0, 1, 2\ldots\} \quad \text{(Eigenfrequencies)}$$

2. Watching their barefoot professor stuff the mouth of a megaphone with his sock makes several of the other students laugh. Dr. Hamilton is not as amused. He takes hold of one of these student's megaphones and stuffs his other sock all the way down to the horn's throat. That is, the throat of the horn is a rigid boundary, and the mouth of the horn is a pressure-release boundary. Find the normal modes and eigenfrequencies of the horn in this case.

   The boundary conditions are now flipped:

$$\left.\frac{\partial p}{\partial x}\right|_{x=0} = 0\ldots\text{rigid end}$$
$$p(L) = 0\ldots\text{pressure release}$$

   As in part (2), the general solution of X is given by

$$X = \exp\left(-\frac{mx}{2}\right)\left(A_1\cos\left(\sqrt{k^2 - \left(\frac{m}{2}\right)^2}x\right) + A_2\sin\left(\sqrt{k^2 - \left(\frac{m}{2}\right)^2}x\right)\right)$$

Applying the boundary condition at $x = L$ shows that $A_2 = 0$. Applying the boundary condition at $x = L$ gives

$$0 = A_1 \cos\left(\sqrt{k^2 - \left(\frac{m}{2}\right)^2} L\right)$$

$$\implies \sqrt{k_n^2 - \left(\frac{m}{2}\right)^2} = \frac{\pi}{2}(2n+1), \; n = \{0, 1, 2 \ldots\} \quad \text{(Equation for } k_n\text{)}$$

The eigenfunctions are therefore

$$p_n(x,t) = \sum_{n=0}^{\infty} B_n \exp\left(-\frac{mx}{2}\right) \cos\left(\frac{\pi}{2L}(2n+1)x\right) e^{\pm j\omega_n t}$$

The eigenfrequencies f_n are the same as before:

$$f_n = \frac{c_0}{2\pi}\sqrt{\left(\frac{\pi}{L}(n+1)\right) + \left(\frac{m}{2}\right)^2}, \; n = \{0, 1, 2 \ldots\} \quad \text{(Eigenfrequencies)}$$