Cylindrical waveguide, double pressure-release

Consider a waveguide with pressure-release surfaces at $z = 0$ and $z = D$. A vertically oriented cylindrical source of radius a extending from $z = 0$ to $z = D$ pulses radially, i.e.,

$$u(a, \theta, z, t) = u_0 e^{j\omega t} \qquad \text{(Velocity source condition)}$$

Solve the pressure wave equation for this configuration.

Note that Blackstock mentions this kind of waveguide on page 431 but solves the case for rigid boundaries at $z = 0$ and $z = D$. The case of a rigid boundary at $z = 0$ and a pressure-release boundary at $z = D$, which is used as an elementary model of sound in the ocean, is dealt with in problem 12-13.

The waves are outgoing, so Hankel functions of the second kind are used. There is no θ-dependence, so $m = 0$. The form of solution reads

$$p(r, z, t) = H_0^{(2)}(\beta r) \begin{Bmatrix} \cos k_z z \\ \sin k_z z \end{Bmatrix} e^{j\omega t} \qquad (1)$$

where $k^2 = \beta^2 + k_z^2$.

The pressure must vanish at $z = 0$, so the $\cos k_z z$ term is thrown out. The pressure must also vanish at $z = D$, so $\sin k_z D = 0$, or

$$k_{z,n} = \frac{n\pi}{D}, \quad n = 1, 2, \ldots$$

Equation (1) becomes

$$p(r, z, t) = \sum_{n=1}^{\infty} A_n H_0^{(2)}(\beta_n r) \sin\left(\frac{n\pi z}{D}\right) e^{j\omega t} \qquad \text{(General solution)}$$

The (Velocity source condition) is satisfied by applying the momentum equation to the above and evaluating at $r = a$:

$$u_0 = \frac{1}{jk\rho c_0} \sum_{n=1}^{\infty} \beta_n A_n H_1^{(2)}(\beta_n a) \sin\left(\frac{n\pi z}{D}\right)$$

The coefficients A_n are then found by the orthogonality of sines:

$$jk\rho_0 c_0 u_0 \int_0^D \sin\left(\frac{m\pi z}{D}\right) \mathrm{d}z = \sum_{n=1}^{\infty} \beta_n A_n H_1^{(2)}(\beta_n a) \int_0^D \sin\left(\frac{n\pi z}{D}\right) \sin\left(\frac{m\pi z}{D}\right) \mathrm{d}z$$

Solving the above for A_n,

$$
\begin{aligned}
A_n &= \frac{2jk\rho_0 c_0 u_0}{D\beta_n H_1^{(2)}(\beta_n a)} \int_0^D \sin\left(\frac{n\pi z}{D}\right) \mathrm{d}z \\
&= \frac{4jk\rho_0 c_0 u_0}{\pi n \beta_n H_1^{(2)}(\beta_n a)}, \quad n = 1, 2, \ldots
\end{aligned}
$$

Using these coefficients in the (General solution) gives the particular solution. Note that many modes are excited, while only the lowest mode is excited in case of the rigid-rigid waveguide.